Fill Ups of Straight Lines and Pair of Straight Lines

Q.1. The area enclosed within the curve | x | + | y | = 1 is .... (1981 - 2 Marks)

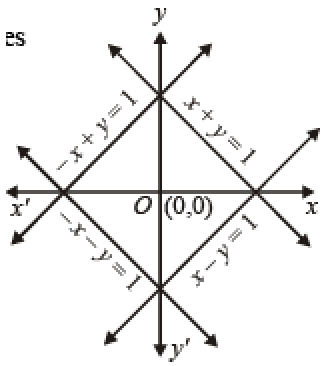
Ans. 2 sq. units.

Sol. $|x| + |y| = 1$

The curve represents four lines

$x + y = 1$, $x - y = 1$, $-x + y = 1$, $-x - y = 1$

which enclose a square of side = distance between opp. sides $x + y = 1$ and $x + y = -1$



$$\text{Side} = \frac{1+1}{\sqrt{1+1}} = \sqrt{2}$$

$\therefore$ Req. area $= (\text{side})^2 = 2$ sq. units.

Q.2. $y = 10^x$ is the reflection of $y = \log 10^{x}$ in the line whose equation is .... (1982 - 2 Marks)

Ans. y = x

Sol. As $y = \log_{10} x$ can be obtained by replacing x by y and y by x in $y = 10^x$

$\therefore$ The line of reflection is $y = x$.

Q.3. The set of lines ax+by+c = 0, where 3a + 2b + 4c = 0 is concurrent at the point .... (1982 - 2 Marks)

Ans. (3/4, 1/2)

Sol.

Given that 3a + 2b + 4c = 0 ⇒ $\frac{3}{4}a+\frac{1}{2}b+c=0$

⇒ The set of lines ax + by + c = 0 passes through the point (3/4, 1/2).

Q.4. Given the points A (0, 4) and B (0, –4), the equation of the locus of the point P(x, y) such that | AP – BP | = 6 is .... (1983 - 1 Mark)

Ans. $\frac{y^2}{9}-\frac{x^2}{7}=1$

Sol. AP – BP | = 6

We know that locus of a point, difference of whose distances from two fixed points is constant, is hyperbola with the fixed points as focii and the difference of distances as length of transverse axis.

Thus, ae = 4 and 2a = 6 ⇒ a = 3, e = 4 /3

$\Rightarrow\ b^2=9\left(\frac{16}{9}-1\right)=7$ ∴ Equation is $\frac{y^2}{9}-\frac{x^2}{7}=1$

(foci being on y-axis, it is vertical hyperbola)

Q.5. If a, b and c are in A.P., then the straight line ax + by + c = 0 will always pass through a fixed point whose coordinates are .... (1984 - 2 Marks)

Ans. (1, – 2)

Sol. If a, b, c are in A.P. then a + c = 2b ⇒ a – 2b + c = 0

⇒ ax + by + c = 0 passes through (1,– 2).

Q.6. The orthocentre of the triangle for med by the lines x + y = 1, 2x + 3y = 6 and 4x – y + 4 = 0 lies in quadrant number .... (1985 - 2 Marks)

Ans. first quadrant

Sol.

The equations of sides of triangle ABC are

AB : $x + y = 1$

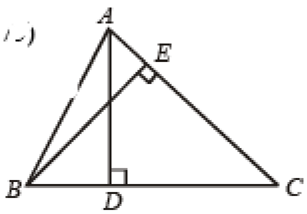
BC : $2x + 3y = 6$

CA : $4x - y = -4$

Solving these pairwise we get the vertices of D as follows A (– 3/5, 8/5) B (– 3, 4) C (– 3/7, 16/7)

Now AD is line ⊥ lar to BC and passes through A. Any line perpendicular to BC is $3x - 2y + l = 0$ As it passes through A(– 3/5, 8/5)

$$\therefore \quad \frac{-9}{5} - \frac{16}{5} + \lambda = 0 \Rightarrow \lambda = 5$$



∴ Equation of altitude AD is $3x - 2y + 5 = 0$ ...(1)

Any line perpendicular to side AC is $x + 4y + \mu = 0$

As it passes through point B (– 3, 4)

$\therefore -3 + 16 + \mu = 0 \Rightarrow \mu = -13$

∴ Equation of altitude BE is $x + 4y - 13 = 0$ ...(2)

Now orthocentre is the point of intersection of equations (1) and (2) (AD and BE)

Solving (1) and (2), we get x = 3/7, y = 22/7

As both the co-ordinates are positive, orthocentre lies in first quadrant.

Q.7. Let the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable straight line be zero; then the line passes through a fixed point whose cordinates are .... (1991 - 2 Marks)

Ans. (1, 1)

Sol. Let the variable line be ax + by + c = 0 .... (1) $\frac{2a+c}{\sqrt{a^2+b^2}} = p_1$

Then $\perp^{lar}$ distance of line from (0, 2) = $\frac{2b+c}{\sqrt{a^2+b^2}} = p_2$

$\perp^{lar}$ distance of line from (1, 1) = $\frac{a+b+c}{\sqrt{a^2+b^2}} = p_3$

ATQ $p_1 + p_2 + p_3 = 0$

$$\frac{2a+c+2b+c+a+b+c}{\sqrt{a^2+b^2}} = 0$$

$\Rightarrow$ 3a + 3b + 3c = 0

$\Rightarrow$ a + b + c = 0 .... (2)

From (1) and (2), we can say variable line (1) passes through the fixed point (1, 1).

Q.8. The vertices of a triangle are A (–1, –7), B (5, 1) and C (1, 4).

The equation of th e bisector of the an gle ∠ABC is .... (1993 - 2 Marks)

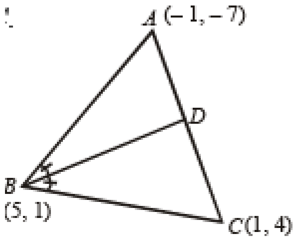
Ans. x - 7y + 2 = 0

Sol. Let BD be the bisector of ∠ABC.

NOTE THIS STEP :

Then AD : DC = AB : BC

And



AB = $\sqrt{(5+1)^2+(1+7)^2}=10$

$BC=\sqrt{(5-1)^2+(1-4)^2}=5$

$\therefore \quad AD:DC=2:1$

∴ By section formula $D\left(\frac{1}{3},\frac{1}{3}\right)$

Therefore equation of BD is

$y-1=\frac{1/3-1}{1/3-5}(x-5) \Rightarrow y-1=\frac{-2/3}{-14/3}(x-5)$

⇒ 7y − 7 = x − 5 ⇒ x − 7y + 2 = 0

True False of Straight Lines and Pair of Straight Lines

Q. 1. The straight line 5x + 4y = 0 passes through the point of intersection of the straight lines x + 2y – 10 = 0 and 2x + y + 5 = 0. (1983 - 1 Mark)
Ans. T

Sol. Intersection point of x + 2y – 10 = 0 and 2x + y + 5 = 0 is $\left(\frac{-20}{3}, \frac{25}{3}\right)$ which clearly satisfies the line 5x + 4y = 0. Hencethe given statement is true.

Q.2. The lines 2x + 3y + 19 = 0 and 9x + 6y – 17 = 0 cut the coordinate axes in concyclic points. (1988 - 1 Mark)

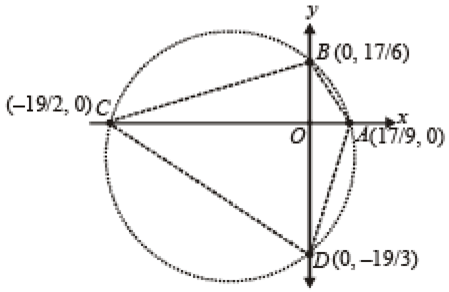
Ans. T

Sol. The given lines cut x-axis at

$A\left(\frac{17}{9}, 0\right), C\left(\frac{-19}{2}, 0\right)$

and y-axis at $B\left(0, \frac{17}{6}\right)$ and $D\left(0, \frac{-19}{3}\right)$.

Now A, B, C, D are concyclic if for AC and BD intersecting at O we have AO × OC = BO × OD



which is true.

∴ The given statement is true.

Subjective questions of Straight Lines and Pair of Straight Lines-1

Q.1. A straight line segment of length l moves with its ends on two mutually perpendicular lines. Find the locus of the point which divides the line segment in the ratio 1 : 2. (1978)

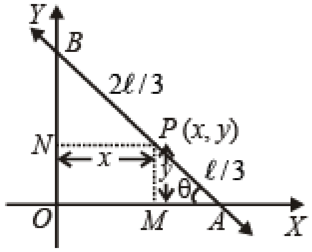
Ans. $9x^2 + 36y^2 = 4\ell^2$

Sol. Let P (x, y) divides line segment AB in the ratio 1 : 2, so that AP = $\ell/3$ and BP = $2\ell/3$ where AB = ℓ .

Then PN = x and PM = y Let ∠ PAM = q = ∠ BPN

In ΔPMA, $\sin\theta = \frac{y}{\ell/3} = \frac{3y}{\ell}$

In ΔPNB, $\cos\theta = \frac{x}{2\ell/3} = \frac{3x}{2\ell}$



Now, $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \frac{9y^2}{\ell^2} + \frac{9x^2}{4\ell^2} = 1 \Rightarrow 9x^2 + 36y^2 = 4\ell^2$$

Q.2. The area of a triangle is 5. Two of its vertices are A (2, 1) and B (3, –2). The third vertex C lies on y = x + 3. Find C. (1978)

Ans. $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(\frac{-3}{2}, \frac{3}{2}\right)$

Sol. As C lies on the line y = x + 3, let the co-ordinates of C be (λ, λ + 3). Also A (2, 1), B (3, – 2).

Then area of ΔABC is given by

$$\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ \lambda & \lambda+3 & 1 \end{vmatrix} = \pm 5$$

$\Rightarrow |2(-2-\lambda-3) - 1(3-\lambda)(3\lambda+9+2\lambda)| = 10$

$\Rightarrow | -2\lambda - 10 - 3 + \lambda + 5\lambda + 9\lambda = 10$

$\Rightarrow | 4\lambda - 4\lambda = 10$

$\Rightarrow 4\lambda - 4 = 10$ or $4\lambda - 4 = -10$

$\Rightarrow \lambda = 7/2$ or $\lambda = -3/2$

∴ Coordinates of C are $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(\frac{-3}{2}, \frac{3}{2}\right)$

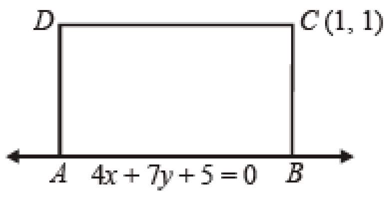
Q.3. One side of a rectangle lies along the line 4x + 7y + 5 = 0. Two of its vertices are (–3, 1) and (1, 1). Find the equations of the other three sides. (1978)

Ans. 4x + 7y -11= 0, 7x - 4y -3 = 0 ; 7x - 4y + 25 = 0

Sol. Let side AB of rectangle ABCD lies along 4x + 7y + 5 = 0.

As (– 3, 1) lies on the line, let it be vertex A.

Now (1, 1) is either vertex C or D.



If (1, 1) is vertex D then slope of AD = 0 ⇒ AD is not perpendicular to AB.

But it is a contradiction as ABCD is a rectangle.
∴ (1, 1) are the co-ordinates of vertex C.
CD is a line parallel to AB and passing through C, therefore equation of CD is

$$y-1=-\frac{4}{7}(x-1)\Rightarrow 4x+7y-11=0$$

Also BC is a line perpendicular to AB and passing through C, therefore equation of BC is

$$y-1=\frac{7}{4}(x-1)\Rightarrow 7x-4y-3=0$$

Similarly, AD is a line perpendicular to AB and passing through A (– 3, 1), therefore equation of line AD is

$$y-1=\frac{7}{4}(x+3)\Rightarrow 7x-4y+25=0$$

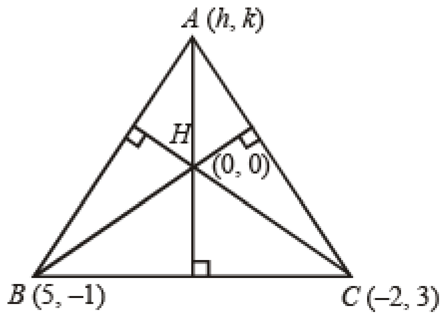
Q.4. (a) Two vertices of a triangle are (5, –1) and (–2, 3). If the orthocentre of the triangle is the origin, find the coordinates of the third point. (b) Find the equation of the line which bisects the obtuse angle between the lines x – 2y + 4 = 0 and 4x – 3y + 2 = 0. (1979)

Ans.

Sol. (a) AH ⊥ BC ⇒ mAH x mBC =-1

$$AH\perp BC\Rightarrow m_{AH}\times m_{BC}=-1$$

$$\Rightarrow\quad \frac{k}{h}\times\frac{3+1}{-2-5}=-1$$



⇒ 4k – 7h = 0 .... (1)

Also, BH ⊥ AC

$$\Rightarrow\quad \frac{-1}{5}\times\frac{3-k}{-2-h}=-1\quad\Rightarrow\quad 3-k=-10-5h$$

⇒ 5h – k + 13 = 0 .... (2)

Solving (1) and (2), we get h = – 4, k = – 7

∴ Third vertex is (– 4, – 7).

(b) The given lines are x – 2y + 4 = 0 .... (1)

and 4x – 3y + 2 = 0 .... (2)

Both the lines have constant terms of same sign.

∴ The equation of bisectors of the angles between the given lines are

$$\frac{x-2y+4}{\sqrt{1+4}}=\pm\frac{4x-3y+2}{\sqrt{16+9}}$$

Here $a_1a_2 + b_1b_2 > 0$ therefore, taking +ve sign on RHS, we get obtuse angle bisector as

$$(4-\sqrt{5})x+(2\sqrt{5}-3)y-(4\sqrt{5}-2)=0 \quad ....(3)$$

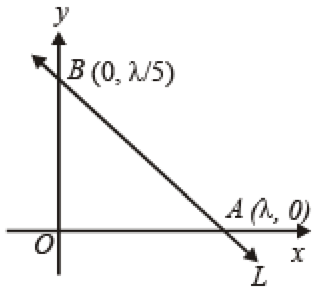
Q.5. A straight line L is perpendicular to the line 5x – y = 1. The area of the triangle formed by the line L and the coordinate axes is 5. Find the equation of the line L. (1980)

Ans.

Sol. The given line is 5x – y = 1

∴ The equation of line L which is perpendicular to the given line is x + 5y = l.

This line meets co-ordinate axes at A (l, 0) and B (0, l/5).



∴ Area of $\Delta OAB=\frac{1}{2}\times OA\times OB$

$\Rightarrow \quad 5 = \frac{1}{2} \times \lambda \times \frac{\lambda}{5}$

$\Rightarrow \lambda^2 = 5^2 \times 2 \Rightarrow \lambda = \pm 5\sqrt{2}$

$\therefore$ The equation of line L is x + 5y − $5\sqrt{2} = 0$

or x + 5y + $5\sqrt{2} = 0$

Q.6. The end A, B of a straight line segment of constant length c slide upon the fixed rectangular axes OX, OY respectively. If the rectangle OAPB be completed, then show that the locus of the foot of the perpendicular drawn from P to AB is

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ **(1983 - 2 Marks)**

Ans.

Sol. From figure,

x = OA – AL

= c cosα – AN cosα

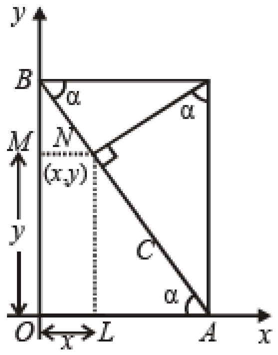
= c cosα – (AP sin a.) cosα

= c cosα – c sinα .

sinα cosα

= c cosα (1 – sin2α)

= c cos3α



y = OB – MB

$y = OB - MB$

$= c \sin\alpha - BN \sin\alpha$

$= c \sin\alpha - BP \cos\alpha \sin\alpha$

$= c \sin\alpha - c \cos\alpha \,.\, \cos\alpha \sin\alpha$

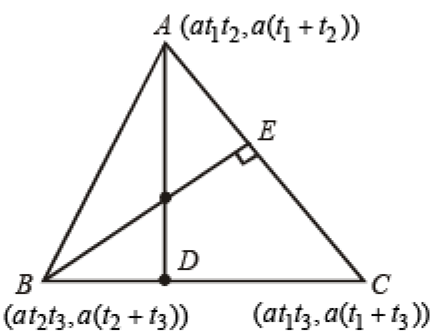
$= c \sin\alpha \,(1 - \cos2\alpha)$

$= c \sin3\alpha$

$\therefore$ Locus of (x, y) is $\left(\frac{x}{c}\right)^{\frac{2}{3}} + \left(\frac{y}{c}\right)^{\frac{2}{3}} = 1$ or $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$

Q.7. The vertices of a triangle are [at_1t_2, $a(t_1 + t_2)$], [at_2t_3, $a(t_2 + t_3)$], [at_3t_1, $a(t_3 + t_1)$]. Find the orthocentre of the triangle. (1983 - 3 Marks)

Ans. Sol.



Slope of BC $= \dfrac{a(t_1+t_3) - a(t_2+t_3)}{at_1t_3 - at_2t_3}$

$= \dfrac{a(t_1+t_3-t_2-t_3)}{at_3(t_1-t_2)} = \dfrac{1}{t_3}$

$\therefore$ Slope of AD $= -\, t_3$

$\therefore$ Eq. of AD,

$$y - a(t_1+t_2) = -t_3(x - at_1t_2)$$
$$\text{or} \quad xt_3 + y = at_1t_2t_3 + a(t_1+t_2) \quad \ldots\ldots\ (1)$$

Similarly, by symm. equation of BE is

$\Rightarrow xt_1 + y = at_1t_2t_3 + a(t_2 + t_3)$ ……. (2)

Solving (1) and (2),

we get $x = -ay = a(t_1 + t_2 + t_3) + at_1t_2t_3)$

$\therefore$ Orthocentre H $(-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$

Q.8. The coordinates of A, B, C are (6, 3), (–3, 5), (4, – 2) respectively, and P is any point (x, y). Show that the ratio of the area of the triangles ΔPBC and ΔABC is $\left|\frac{x+y-2}{7}\right|$ (1983 - 2 Marks)

Ans. Sol. Area of ΔABC $= \frac{1}{2}\begin{vmatrix} 6 & 3 & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$

$$= \frac{1}{2}[6(7) + 3(5) + 4(-2)] = \frac{49}{2}$$

Area of ΔPBC $= \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$

$$= \frac{1}{2}(7x + 7y - 14) - \frac{7}{2}|x + y - 2|$$

$$ATQ, \frac{Ar(\Delta PBC)}{Ar(\Delta ABC)} = \frac{\frac{7}{2}|x+y-2|}{\frac{49}{2}} = \left|\frac{x+y-2}{7}\right|$$

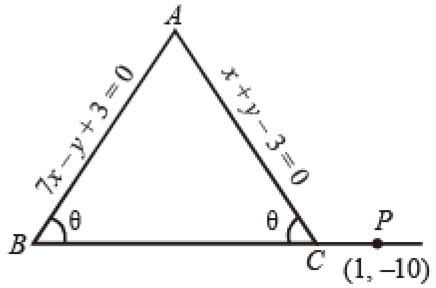
Q.9. Two equal sides of an isosceles triangle are given by the equations 7x -y + 3=0 and x +y - 3=0 and its third side passes through the point (1, –10). Determine the equation of the third side. (1984 - 4 Marks)

Ans. Sol. Let equations of equal sides AB and AC of isosceles ΔABC are

$7x - y + 3 = 0$ …….. (1)

and $x + y - 3 = 0$ …….. (2)

The third side BC of Δ passes through the point (1, – 10). Let its slope be m.



As AB = AC

∴∠B = ∠C

⇒ tan B = tan C .... (3)

Now slope of AB = 7 and slope of AC = – 1

Using $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, we get

$\tan B = \left|\frac{7-m}{1+7m}\right|$ and $\tan C = \left|\frac{-1-m}{1-m}\right|$

From eq. (3), we get

$$\left|\frac{7-m}{1+7m}\right| = \left|\frac{-1-m}{1-m}\right|$$

$$\Rightarrow \frac{7-m}{1+7m} = \pm\left(\frac{-1-m}{1-m}\right)$$

Taking '+ ' sign, we get (7 – m) (1 – m) = – (1+ m) (1 + 7m)

⇒ 7 – 8m + m^2 + $7m^2$ + 8m + 1 = 0 ⇒ $8m^2$ + 8 = 0

⇒ m^2 + 1 = 0

It has no real solution.

Taking '–' sign, we get (7 – m) (1 – m) = (1 + m) (1 + 7m)

⇒ 7 – 8m + m^2 – $7m^2$ – 8m – 1 = 0

$\Rightarrow -6m^2 - 16m + 6 = 0$

$\Rightarrow 3m^2 + 8m - 3 = 0$

$\Rightarrow (3m - 1)(m + 3) = 0$

$\Rightarrow m = 1/3, -3$

$\therefore$ The required line is

$y + 10 = \frac{1}{3}(x - 1)$ or $y + 10 = -3(x - 1)$

i.e. $x - 3y - 31 = 0$ or $3x + y + 7 = 0$.

Q.10. One of the diameters of the circle circum scribing the rectangle ABCD is 4 y = x + 7 . If A and B are the points (–3, 4) and (5, 4) respectively, then find the area of rectangle. (1985 - 3 Marks)

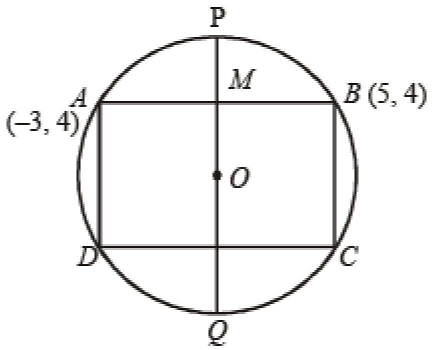
Ans. 32 sq. units

Sol. Let O be the centre of the circle. M is the mid point of AB.

Then

OM $\perp$ AB

Let OM when produced meets the circle at P and Q.



$\therefore$ PQ is a diameter perpendicular to AB and passing through M.

$$M = \left(\frac{-3+5}{2}, \frac{4+4}{2}\right) = (1, 4)$$

Slope of AB $= \frac{4-4}{5+3} = 0$

∴ PQ, being perpendicular to AB, is a line parallel to yaxis passing through (1, 4).

∴ Its equation is x = 1

……. (1)

Also eq. of one of the diameter given is 4y = x + 7 ……. (2)

Solving (1) and (2), we get co-ordinates of centre OO(1, 2)

Also let co-ordinates of D be (α, b)

Then O is mid point of BD, therefore

$$\left(\frac{\alpha+5}{2}, \frac{\beta+4}{2}\right) = (1, 2) \Rightarrow a = -3, b = 0$$

∴ D (– 3, 0)

Using the distance formula we get

$$AD = \sqrt{(-3+3)^2 + (4-0)^2} = 4$$

$$AB = \sqrt{(5+3)^2 + (4-4)^2} = 8$$

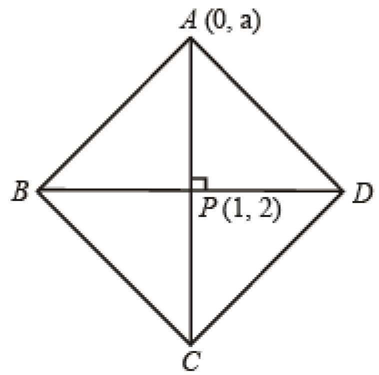
∴ Area of rectangle ABCD = AB × AD = 8 × 4 = 32 square units.

Q.11. Two sides of a rh ombus ABCD ar e par allel to the lines y = x + 2 an d y = 7 x + 3 . If th e diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on the y-axis, fin d possible co-ordinates of A. (1985 - 5 Marks)

Ans. (0, 0) or (0, 5/2)

Sol. A being on y-axis, may be chosen as (0, a).

The diagonals intersect at P (1, 2).



Again we know that diagonals will be parallel to the angle bisectors of the two sides y = x + 2 and y = 7x + 3

i.e., $\frac{x-y+2}{\sqrt{2}} = \pm\frac{7x-y+3}{5\sqrt{2}}$

$\Rightarrow 5x - 5y + 10 = \pm (7x - y + 3)$

$\Rightarrow 2x + 4y - 7 = 0$ and $12x - 6y + 13 = 0$

$m_1 = -1/2$

$m_2 = 2$

Let diagonal d_1 be parallel to $2x + 4y - 7 = 0$ and diagonal d_2 be parallel to $12x - 6y + 13 = 0$.

The vertex A could be on any of the two diagonals. Hence slope of AP is either $-1/2$ or 2.

$\Rightarrow \frac{2-a}{1-0} = 2$ or $\frac{-1}{2}$

$\Rightarrow a = 0$ or 5 $\frac{5}{2}$

$\therefore$ A is $(0, 0)$ or $(0, 5/2)$

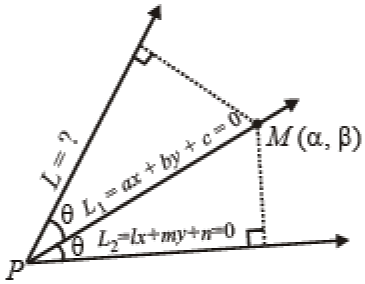
Q.12. Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at the point P and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1. (1988 - 5 Marks)

Ans. $(a^2 + b^2)(lx + my + n) - 2(al + bm)(ax + by + c) = 0$

Sol. Let the equation of other line L, which passes through the point of intersection P of lines $L_1 \equiv ax + by + c = 0$ .... (1)

and $L_2 \equiv \ell x + my + n = 0$ .... (2)

be $L_1 + \lambda L_2 = 0$ i.e. $(ax + by + c) + \lambda(\ell x + my + n) = 0$ .... (3)



From figure it is clear that L_1 is the bisector of the angle between the lines given by (2) and (3) [i.e. L_2 and L] Let M (α, β) be any point on L_1 then $a\alpha + b\beta + c = 0$ ...... (4)

Also from M, lengths of perpendiculars to lines L and L_2 given by equations (3) and (4), are equal

$$\frac{\ell\alpha + m\beta + n}{\sqrt{\ell^2 + m^2}} = \pm\frac{(a\alpha + b\beta + c) + \lambda(l\alpha + m\beta + n)}{\sqrt{(a+\lambda)^2 + (b+\lambda m)^2}}$$

$$\Rightarrow \frac{1}{\sqrt{\ell^2 + m^2}} = \pm\frac{\lambda}{\sqrt{(\ell^2 + m^2)\lambda^2 + 2(a\ell + bm)\lambda + (a^2 + b^2)}}$$

[Using 4]

$$\Rightarrow (\ell^2 + m^2)\lambda^2 + 2(a\ell + bm)\lambda + (a^2 + b^2) = \lambda^2(\ell^2 + m^2)$$

$$\Rightarrow \lambda = -\frac{a^2 + b^2}{2(a\ell + bm)}$$

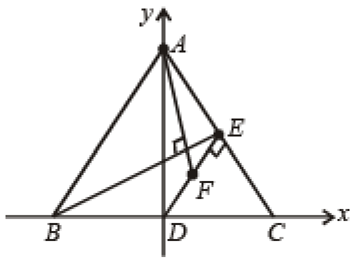
Substituting this value of l in eq. (3), we get L as

$$(ax + by + c) - \frac{(a^2 + b^2)}{2(a\ell + bm)}(\ell x + my + n) = 0$$

$\Rightarrow (a^2 + b^2)(\ell x + my + n) - 2(a\ell + bm)(ax + by + c) = 0$

Q.13. Let ABC be a triangle with AB = AC. If D is the midpoint of BC, E is the foot of the perpendicular drawn from D to AC and F the mid-point of DE, prove that AF is perpendicular to BE. (1989 - 5 Marks)

Ans. Sol. Let BC be taken as x-axis with origin at D, the mid-point of BC, and DA will be y-axis.



AB = AC.

Let BC = 2a, then the co-ordinates of B and C are (– a, 0) and (a, 0).

Let DA = h, so that co-ordinates of A are (0, h).

Then equation of AC is $\frac{x}{a}+\frac{y}{h}=1$ .... (1)

And equation of DE ⊥ to AC and passing through origin is

$\frac{x}{h}-\frac{y}{a}=0 \Rightarrow x=\frac{hy}{a}$........ (2)

Solving (1) and (2) we get the co-ordinates of pt E as follows

$\frac{hy}{a^2}+\frac{y}{h}=1$ ⇒ h2 y + a2 y = a2 h

$\Rightarrow \quad y=\frac{a^2h}{a^2+h^2} \quad \Rightarrow \quad x=\frac{ah^2}{a^2h^2}$

$\therefore \quad E\left(\frac{ah^2}{a^2+h^2}, \frac{a^2h}{a^2+h^2}\right)$

Since F is mid pt. of DE, therefore, its co-ordinates are

$F\left(\frac{ah^2}{2(a^2+h^2)}, \frac{a^2h}{2(a^2+h^2)}\right)$

$$\therefore \text{Slope of AF} = \frac{h - \frac{a^2 h}{2(a^2+h^2)}}{0 - \frac{ah^2}{2(a^2+h^2)}} = \frac{2h(a^2+h^2) - a^2 h}{-ah^2}$$

$$\Rightarrow m_1 = -\frac{a^2+2h^2}{ah} \text{........ (i)}$$

$$\text{And slope of BE} = \frac{\frac{a^2 h}{a^2+h^2} - 0}{\frac{ah^2}{a^2+h^2} + a} = \frac{a^2 h}{ah^2 + a^3 + ah^2}$$

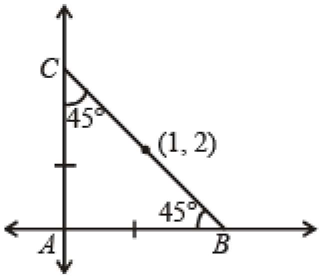
$$\Rightarrow m_2 = \frac{ah}{a^2+2h^2} \text{....... (ii)}$$

From (i) and (ii), we observe that $m_1 m_2 = -1 \Rightarrow AF \perp BE$. Hence Proved.

Q.14. Straight lines 3x + 4y = 5 and 4x – 3y = 15 intersect at the point A. Points B and C are chosen on these two lines such that AB = AC. Determine the possible equations of the line BC passing through the point (1, 2). (1990 - 4 Marks)

Ans. Sol. The given st. lines are 3x + 4y = 5 and 4x – 3y = 15.

Clearly these st. lines are perpendicular to each other ($m_1 m_2 = -1$), and intersect at A. Now B and C are pts on these lines such that AB = AC and BC passes through (1, 2) From fig. it is clear that ∠B = ∠C = 45°



Let slope of BC be m. Then using

$$\tan B = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \text{, we get } \tan 45° = \left|\frac{m + 3/4}{1 - \frac{3}{4}m}\right|$$

$\Rightarrow 4m + 3 = \pm (4 - 3m)$

$\Rightarrow 4m + 3 = 4 - 3m$ or $4m + 3 = -4 + 3m$

$\Rightarrow m = 1/7$ or $m = -7$

$\therefore$ Eq. of BC is, $y - 2 = \frac{1}{7}(x - 1)$

or $y - 2 = -7(x - 1)$

$\Rightarrow 7y - 14 = x - 1$ or $y - 2 = -7x + 7$

$\Rightarrow x - 7y + 13 = 0$ or $7x + y - 9 = 0$

Subjective questions of Straight Lines and Pair of Straight Lines-2

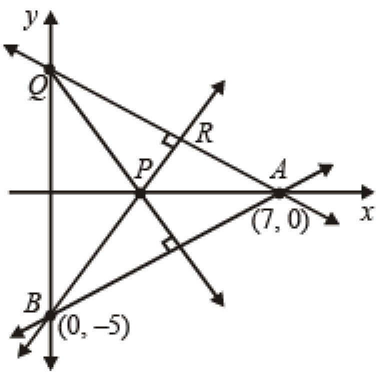
Q.15. A line cuts the x-axis at A (7, 0) and the y-axis at B(0, – 5). A variable line PQ is drawn perpendicular to AB cutting the xaxis in P and the y-axis in Q. If AQ and BP intersect at R, find the locus of R. (1990 - 4 Marks)

Ans. $x^2 + y^2 - 7x + 5y = 0$

Sol. Eq. of the line AB is

$$\frac{x}{7} - \frac{y}{5} = 1 \quad [A\,(7, 0), B(0, -5)]$$

$\Rightarrow 5x - 7y - 35 = 0$ Eq. of line PQ ⊥ AB is $7x + 5y + l = 0$ which meets axes of x and y at pts P(– l/7, 0) and Q (0, – l/5) resp.



Eq. of AQ is,

$$\frac{x}{y} + \frac{y}{-\lambda/5} = 1 \Rightarrow \lambda x - 35y - 7\lambda = 0 \; .... \; (2)$$

Eq. of BP is,

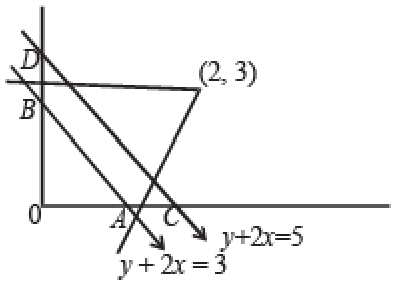
$$\frac{-7x}{\lambda} - \frac{y}{5} = 1 \Rightarrow 35x + \lambda y + 5\lambda = 0 \; .... \; (3)$$

Locus of R the pt. of intersection of (2) and (3) can be obtained by eliminating l from these eq. ' s, as follows

$$35x + (5 + y)\left(\frac{35y}{x-7}\right) = 0$$

$\Rightarrow 35x\,(x - 7) + 35y\,(5 + y) = 0 \;\Rightarrow x^2 + y^2 - 7x + 5y = 0$

Q.16. Find the equation of the line passing through the point (2, 3) and making intercept of length 2 units between the lines y + 2x = 3 and y + 2x = 5. (1991 - 4 Marks)



Ans. 3x + 4y -18 = 0 or x - 2 = 0

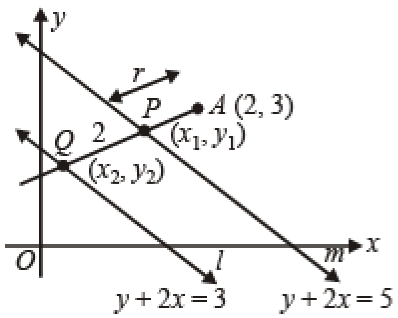
Sol. Let the equation of line through A which makes an intercept of 2 units between.

2x + y = 3 .... (1)

and 2x + y = 5 .... (2)

be $\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r$

Let AP = r then AQ = r + 2



Then for pt P (x_1, y_1),

$$\frac{x_1-2}{\cos\theta} = \frac{y_1-3}{\sin\theta} = r \;\Rightarrow\; \frac{2(x_1-2)+(y_1-3)}{2\cos\theta+\sin\theta} = r$$

$$\left(\text{Using } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{\lambda a_1 + \mu b_1}{\lambda a_2 + \mu b_2}\right)$$

$\Rightarrow \dfrac{(2x_1+y_1)-7}{2\cos\theta+\sin\theta}=r \quad \Rightarrow \dfrac{5-7}{2\cos\theta+\sin\theta}=r$

[Using $2x_1 + y_1 = 5$ as P (x_1, y_1) lies on $2x + y = 5$]

$\dfrac{-2}{2\cos\theta+\sin\theta}=r$ ... (i)

For pt Q (x_2, y_2),

$\dfrac{x_2-2}{\cos\theta}=\dfrac{y_2-3}{\sin\theta}=r+2$

$\Rightarrow \dfrac{2(x_2-2)+(y_2-3)}{2\cos\theta+\sin\theta}=r+2$

$\Rightarrow \dfrac{-4}{2\cos\theta+\sin\theta}=r+2$ ...(ii)

(ii) – (i)

$\Rightarrow \dfrac{-2}{2\cos\theta+\sin\theta}=2$

$\Rightarrow 2\cos\theta + \sin\theta = -1$ … (3)

$\Rightarrow 2\cos\theta = -(1 + \sin\theta)$

Squaring on both sides, we get

$\Rightarrow 4\cos^2\theta = 1 + 2\sin\theta + \sin^2\theta$

$\Rightarrow (5\sin\theta - 3)(\sin\theta + 1) = 0$

$\Rightarrow \sin\theta = 3/5, -1$

$\Rightarrow \cos\theta = -4/5, 0$ [Using eq. (3)]

∴ The required equation is either

$\dfrac{x-2}{-4/5}=\dfrac{y-3}{3/5}$ or $\dfrac{x-2}{0}=\dfrac{y-3}{-1}$

$\Rightarrow$ either $3x - 6 = -4y + 12$ or $x - 2 = 0$

$\Rightarrow$ either $3x + 4y - 18 = 0$ or $x - 2 = 0$

Q.17. Show that all chords of the curve 3 x^2 -y^2 – 2x + 4y = 0, which subtend a right angle at the origin, pass through a fixed point. Find the coordinates of the point. (1991 - 4 Marks)

Ans. (1, –2)

Sol. The given curve is $3x^2 - y^2 - 2x + 4y = 0$ … (1)

Let y = mx + c be the chord of curve (1) which subtends an ∠ of 90° at origin.
Then the combined eq. of lines joining points of intersection of curve (1) and chord y = mx + c to the origin, can be obtained by making the eq. of curve homogeneous with the help of eq. of chord, as follows.

$$3x^2 - y^2 - 2x\left(\frac{y-mx}{c}\right) + 4y\left(\frac{y-mx}{c}\right) = 0$$

$\Rightarrow (3c + 2m)\, x^2 - 2\,(1+ 2m)\, xy + (4 - c)\, y^2 = 0$

As the lines represented by this pair are perpendicular to each other, therefore we must have coeff. of x^2+ coeff. of $y^2 = 0$

$\Rightarrow 3c + 2m + 4 - c = 0 \Rightarrow -2 = m \cdot 1 + c$

Which on comparision with eq. of chord, implies that y = mx + c passes though (1, – 2).

Hence the family of chords must pass through (1, – 2).

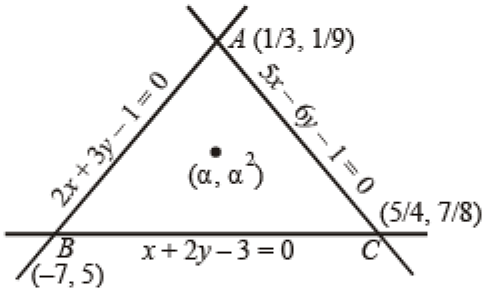
Q.18. Determine all values of a for which the point (α, α^2) lies inside the triangle formed by the lines 2x + 3y – 1 = 0 (1992 - 6 Marks)

x + 2y – 3 = 0
5x – 6y – 1 = 0

Ans. $\alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$

Sol. The points of intersection of given lines are

$$A\left(\frac{1}{3}, \frac{1}{9}\right), B(-7, 5), C\left(\frac{5}{4}, \frac{7}{8}\right)$$



If (α, α^2) lies inside the Δ formed by the given lines, then

$\left(\frac{1}{3},\frac{1}{9}\right)$ and (α, α^2) lie on the same side of the line $x + 2y - 3 = 0$

$$\Rightarrow \frac{\alpha+2\alpha^2-3}{\frac{1}{3}+\frac{2}{9}-3} > 0 \Rightarrow 2\alpha^2+\alpha-3<0 \ldots (1)$$

Similarly $\left(\frac{5}{4},\frac{7}{8}\right)$ and (α, α^2) lie on the same side of the line $2x + 3y - 1 = 0$.

$$\Rightarrow \frac{2\alpha+3\alpha^2-1}{\frac{10}{4}+\frac{21}{8}-1} > 0 \Rightarrow 3\alpha^2+2\alpha-1>0 \ldots (2)$$

$(-7, 5)$ and (α, α^2) lie on the same side of the line $5x - 6y - 1 = 0$.

$$\Rightarrow \frac{5\alpha+6\alpha^2-1}{-35-30-1} > 0 \Rightarrow 6\alpha^2-5\alpha+1>0 \ldots (3)$$

$$\therefore \quad \alpha \in \left(-\frac{3}{2},-1\right)\cup\left(\frac{1}{2},1\right)$$

Q.19. Tagent at a point P_1 {other than (0, 0)} on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissae of P_1 , P_2 , P_3 ....P_n , form a G.P. Also find the ratio. [area (DP_1 , P_2 , P_3)] /[area (P_2 P_3 ,P_4)] (1993 - 5 Marks)

Ans.

Sol. The given curve is

$y = x^3 \quad \ldots (1)$

Let the pt, P_1 be (t, t^3), $t \neq 0$\

Then slope of tangent at $P_1 = \frac{dy}{dx} = (3x^2)_{x=t} = 3t^2$

$\therefore$ Equation of tangent at P_1 is $y - t^3 = 3t^2 (x - t)$

$\Rightarrow y = 3t^2 x - 2t^3 \Rightarrow 3t^2x - y - 2t^3 = 0 \ldots (2)$

Now this tangent meets the curve again at P_2 which can be obtained by solving (1) and (2) i.e., $3t^2x - x^2 - 2t^3 = 0$

or $x^3 - 3t^2x + 2t^3 = 0$ $(x - t)^2 (x + 2t) = 0$

$\Rightarrow x = -2t$ as $x = t$ is for P_1

$\therefore y = -8t^3$ Hence pt P_2 is $(-2t, -8t^2) = (t_1, t_1^3)$s ay..

Similarly, we can find that tangent at P_2 which meets the curve again at P_3 $(2t_1, -8t_1^3)$ i.e., $(4t, 64t^3)$.

Similarly, $P_4 \equiv (-8t, -512t^3)$ and so on.

We observe that abscissae of pts. $P_1, P_2, P_3\ldots$ are $t, -2t, 4t, \ldots$ which form a GP with common ratio -2.

Also ordinates of these pts. $t^3, -8t^3, 64t^3, \ldots$ also form a GP with common ratio -8. Now,

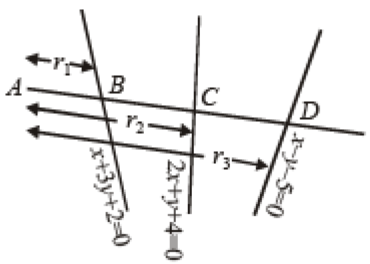
$$\text{Now, } \frac{Ar(\Delta P_1P_2P_3)}{Ar(\Delta P_2P_3P_4)} = \frac{\begin{vmatrix} 1 & t & t^3 \\ 1 & -2t & -8t^3 \\ 1 & 4t & 64t^3 \end{vmatrix}}{\begin{vmatrix} 1 & -2t & -8t^3 \\ 1 & 4t & -64t^3 \\ 1 & -8t & -512t^3 \end{vmatrix}}$$

$$= \frac{t^4\begin{vmatrix}1 & 1 & 1\\ 1 & -2 & -8\\ 1 & 4 & 64\end{vmatrix}}{(-2)(-8)t^4\begin{vmatrix}1 & 1 & 1\\ 1 & -2 & -8\\ 1 & 4 & -64\end{vmatrix}} = \frac{1}{16} \text{ sq. units.}$$

Q.20. A line through A (–5, –4) meets the line x + 3y + 2 = 0, 2x + y + 4 = 0 and x – y – 5 = 0 at the points B, C and D respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line. (1993 - 5 Marks)

Ans. 2x + 3y + 22 = 0

Sol. Let θ be the inclination of line through A (– 5, – 4). Therefore equation of this line is



$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_1, r_2, r_3$$

⇒ B (r_1 cosθ– 5, r_1 sinθ – 4)

C (r_2 cosθ – 5, r_2 sinθ – 4)

D (r_3 cosθ – 5, r_3 sinθ – 4)

But B lies on x + 3y + 2 = 0. therefore r_1 cosθ – 5 + 3r_1 sin θ – 12 + 2 = 0

$$\Rightarrow \quad r_1 = \frac{15}{\cos\theta + 3\sin\theta} = AB$$

$$\Rightarrow \quad \frac{15}{AB} = \cos\theta + 3\sin\theta \quad \ldots (1)$$

As C lies on 2x + y + 4 = 0, therefore 2 (r_2 cosθ – 5) + (r_2 sinθ – 4) + 4 = 0

$\Rightarrow \quad r_2 = \dfrac{10}{2\cos\theta + \sin\theta} = AC$

$\Rightarrow \quad \dfrac{10}{AC} = 2\cos\theta + \sin\theta$ ... (2)

Similarly D lines on x – y – 5 = 0, therefore $r_3 \cos\theta - 5 - r_3 \sin\theta + 4 - 5 = 0$

$\Rightarrow \quad r_3 = \dfrac{6}{\cos\theta - \sin\theta} = AD$

$\Rightarrow \quad \dfrac{6}{AD} = \cos\theta - \sin\theta$ ... (3)

Now, ATQ, $\left(\dfrac{15}{AB}\right)^2 + \left(\dfrac{10}{AC}\right)^2 = \left(\dfrac{6}{AD}\right)^2$

$\Rightarrow (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$ [Using (1), (2) and (3)]

$\Rightarrow 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$

$\Rightarrow 2\cos\theta + 3\sin\theta = 0$

$\Rightarrow \quad \tan\theta = -\dfrac{2}{3}$

$\therefore$ Equation of req. line is $y + 4 = -\dfrac{2}{3}(x+5)$

$\Rightarrow \quad 2x + 3y + 22 = 0$

Q.21. A rectangle PQRS has its side PQ parallel to the line y = mx and vertices P, Q and S on the lines y = a, x = b and x = –b, respectively. Find the locus of the vertex R. (1996 - 2 Marks)

Ans. $x(m^2 - 1) - ym + (m^2 + 1)b + am = 0$

Sol. Let the co-ordinates of Q be (b, a) and that of S be (– b, b).

Let PR and SQ intersect each other at G.

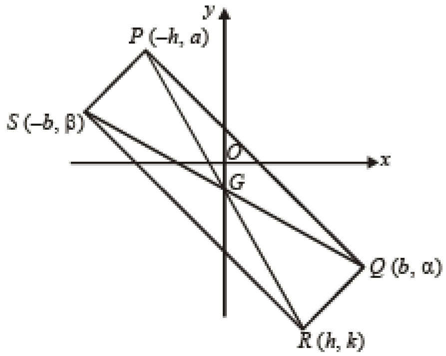
$\because$ G is the mid pt of SQ. ($\because$ diagonals of a rectangle bisect each other)

$\therefore$ x co-ordinates of G must be a.

Let the co-ordinates of R be (h, k).

$\therefore$ The x-coordinates of P is – h ($\because$ G is the mid point of PR)

As P lies on y = a, therefore cordinates of P are (– h, a).



$\because$ PQ is parallel to y = mx, Slope of PQ = m

$\therefore \quad \frac{\alpha - a}{b+h} = m \Rightarrow \alpha = a + m(b+h) \ldots (1)$

Also RQ $\perp$ PQ$\Rightarrow$

Slope of $RQ = \frac{-1}{m}$

$\therefore \quad \frac{k-\alpha}{h-b} = \frac{-1}{m} \Rightarrow \alpha = k + \frac{1}{m}(h-b) \cdots (2)$

From (1) and (2) we ge

$a + m(b+h) = k + \frac{1}{m}(h-b)$

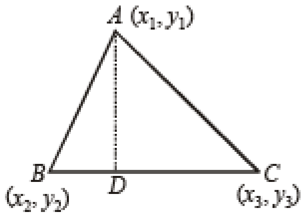
$\Rightarrow (m^2 - 1)h - mk + b(m^2 + 1) + am = 0$

$\therefore$ Locus of vertex R (h, k) is $(m^2 - 1)x - my + b(m^2 + 1) + am = 0$.

Q.22. Using co-ordinate geometry, prove that the three altitudes of any triangle are concurrent. (1998 - 8 Marks)

Ans.

Sol. Let A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) be the vertices of ΔABC



Then equation of alt. AD is

$$y - y_1 = -\left[\frac{x_2 - x_3}{y_2 - y_3}\right](x - x_1)$$

or$(x - x_1)(x_2 - x_3) + (y - y_1)(y_2 - y_3) = 0$ ... (1)

Similarly equations of other two attitudes are

$(x - x_2)(x_3 - x_1) + (y - y2)(y_3 - y_1) = 0$ ...(2)

and $(x - x_3)(x_1 - x_2) + (y - y_3)(y_1 - y_2) = 0$ ...(3)

Now, above three lines will be concurrent if

$$\begin{vmatrix} x_2 - x_3 & y_2 - y_3 & -x_1(x_2 - x_3) - y_1(y_2 - y_3) \\ x_3 - x_1 & y_3 - y_1 & -x_2(x_3 - x_1) - y_2(y_3 - y_1) \\ x_1 - x_2 & y_1 - y_2 & -x_3(x_1 - x_2) - y_3(y_1 - y_2) \end{vmatrix} = 0$$

On L.H.S.

Operating $R_1 + R_2 + R_3$, R_1 becomes row of zeros.

∴ Value of determinant = 0 = R.H.S.

Hence the altitudes are concurrent.

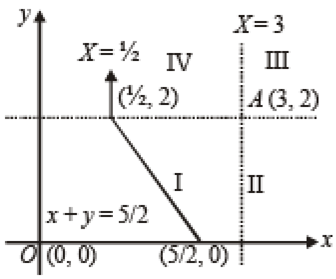
Q.23. For points P = (x 1, y 1) an d Q = (x_2, y_2) of the co-ordinate plane, a new distance d(P, Q) is defined by d(P, Q) = $|x_1 - x_2| + |y_1 - y_2|$. Let O = (0, 0) and A = (3, 2). Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram. (2000 - 10 Marks)

Ans.

Sol. Let P = (h, k) be a general point in the first quadrant such that d (P, A) = d (P, O)

$\Rightarrow |h - 3| + |k - 2| = |h| + |k| = h + k$ ... (1)

[h and k are + ve, pt (h, k) being in I quadrant.]



If h < 3, k < 2 then (h, k) lies in region I.

It h > 3, k < 2, (h, k) lies in region II.

If h > 3, k > 2 (h, k) lies in region III.

If h < 3, k > 2 (h, k) lies in region IV.

In region I, eq. (1)

$\Rightarrow 3 - h + 2 - k = h + k \Rightarrow h + k = \frac{5}{2}$

In region II, eq. (1) becomes

$\Rightarrow h - 3 + 2 - k = h + k \Rightarrow k = -\frac{1}{2}$ not possible.

In region III, eq. (1) becomes

$\Rightarrow h - 3 + k - 2 = h + k \Rightarrow - 5 = 0$ not possible.

In region IV, eq. (1) becomes

$\Rightarrow h - 3 + k - 2 = h + k \Rightarrow - 5 = 0$ not possible.

In region IV, eq. (1) becomes

$\Rightarrow 3 - h + k - 2 = h + k \Rightarrow h = 1/2$

⇒ Hence required set consists of line segment x + y = 5/2 of finite length as shown in the first region and the ray x = 1/2 in the fourth region.

Q.24. Let ABC and PQR be any two triangles in the same plane. Assume that the prependiculars from the points A, B, C to the sides QR, RP, PQ respectively are concurrent. Using vector methods or otherwise, prove that the prependiculars from P, Q, R to BC, CA, AB respectively are also concurrent. (2000 - 10 Marks)

Ans.

Sol. Let the co-ordinates of the vertices of the ΔABC be A (a_1, b_1), B(a_2, b_2) and C (a_3, b_3) and co-ordinates of the vertices of the ΔPQR be P (x_1, y_1), B (x_2, y_2) and R (x_3, y_3)

Slope of $QR = \frac{y_2 - y_3}{x_2 - x_3}$

⇒ Slope of straight line perpendicular to

$QR = -\frac{x_2 - x_3}{y_2 - y_3}$

Equation of straight line passing through A (a_1, b_1) and perpendicular to QR is

$$y - b_1 = -\frac{x_2 - x_3}{y_2 - y_3}(x - a_1)$$

⇒ $(x_2 - x_3)x + (y_2 - y_3)\,y - a_1\,(x_2 - x_3) - b_1\,(y_2 - y_3) = 0$ … (1)

Similarly equation of straight line from B and perpendicular to RP is $(x_3 - x_1)\,x + (y_3 - y_1)\,y - a_2\,(x_3 - x_1) - b_2\,(y_3 - y_1) = 0$ … (2)

and eqn of straight line from C and perpendicular to PQ is $(x_1 - x_2)\,x + (y_1 - y_2)\,y - a_3\,(x_1 - x_2) - b_3(y_1 - y_2) = 0$ … (3)

As straight lines (1), (2) and (3) are given to be concurrent, we should have

$$\begin{vmatrix} x_2 - x_3 & y_2 - y_3 & a_1(x_2 - x_3) + b_1(y_2 - y_3) \\ x_3 - x_1 & y_3 - y_1 & a_2(x_3 - x_1) + b_2(y_3 - y_1) \\ x_1 - x_2 & y_1 - y_2 & a_1(x_1 - x_2) + b_3(y_1 - y_2) \end{vmatrix} = 0 \quad \ldots (4)$$

Operating $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 0 & 0 & S \\ x_3-x_1 & y_3-y_1 & a_2(x_3-x_1)+b_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & a_3(x_1-x_2)+b_3(y_1-y_2) \end{vmatrix} = 0$$

where

$$[S = a_1(x_2-x_3)+b_1(y_2-y_3)+a_2(x_3-x_1)+b_2(y_3-y_1)+a_3(x_1-x_2)+b_3(y_1-y_2)]$$

Expanding along R_1

$$\Rightarrow [(x_3-x_1)(y_1-y_2)-(x_1-x_2)(y_3-y_1)]\, S = 0$$

$$\Rightarrow \left[\frac{y_1-y_2}{x_1-x_2}-\frac{y_3-y_1}{x_3-x_1}\right] S = 0$$

$$\Rightarrow [m_{PQ}-m_{PR}]\, S=0 \Rightarrow S=0$$
$$[m_{PQ}=m_{PR} \Rightarrow PQ \parallel PR$$

which is not possible in ΔPQR]

$\Rightarrow a_1(x_2-x_3)+b_1(y_2-y_3)+a_2(x_3-x_1)+b_2(y_3-y_1)+a_3(x_1-x_2)+b_3(y_1-y_2)=0$ … (5)

$\Rightarrow x_1(a_3-a_2)+y_1(b_3-b_2)+x_2(a_1-a_3)+y_2(b_1-b_3)+x_3(a_2-a_1)+y_3(b_2-b_1)=0$ ...(6)

(Rearranging the equation (5)) But above condition shows

$$\begin{vmatrix} a_3-a_2 & b_3-b_2 & x_1(a_3-a_2)+y_1(b_3-b_2) \\ a_1-a_3 & b_1-b_3 & x_2(a_1-a_3)+y_2(b_1-b_3) \\ a_2-a_1 & b_2-b_1 & x_3(a_2-a_1)+y_3(b_2-b_1) \end{vmatrix} = 0 \quad ...(7)$$

[Using the fact that as (4) ⇔ (5) in the same way (6) ⇔ (7)]

Clearly equation (7) shows that lines through P and perpendicular to BC, through Q and perpendicular to AB are concurrent. Hence Proved.

Q.25. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that

the equation $\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$

represents a straight line. (2001 - 6 Marks)

Ans.

Sol. $C_1 \to aC_1$

$$\Delta = \frac{1}{a}\begin{vmatrix} a^2x-aby-ac & bx+ay & cx+a \\ abx+a^2y & -ax+by-c & cy+b \\ acx+a^2 & cy+b & -ax-by+c \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$\Delta = \frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & ay+bx & cx+a \\ (a^2+b^2+c^2)y & by-c-ax & cy+b \\ (a^2+b^2+c^2) & b+cy & -ax-by+c \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} x & ay+bx & cx+a \\ y & by-c-ax & b+cy \\ 1 & b+cy & c-ax-by \end{vmatrix},$$

as $a^2 + b^2 + c^2 = 1$

$C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$

$$\text{then } \Delta = \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

$$= \frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

$R_1 \to R_1 + yR_2 + R_3$

$$\Delta = \frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

On expanding along R_1

$$\Delta = \frac{(x^2+y^2+1)}{ax} ax(ax+by+c)$$

$= (x_2 + y_2 + 1)\,(ax + by + c)$

Given $\Delta = 0$

$\Rightarrow ax + by + c = 0$, which represents a straight line.

$[\because x_2 + y_2 + 1 \neq 0$, being + ve].

Q.26. A straight line L through the origin meets the lines x + y = 1 and x + y = 3 at P and Q respectively. Through P and Q two straight lines L_1 and L_2 are drawn, parallel to 2x – y = 5 and 3x + y = 5 respectively. Lines L_1 and L_2 intersect at R. Show that the locus of R, as L varies, is a straight line. (2002 - 5 Marks)

Sol. The line y = mx meets the given lines in

$P\left(\frac{1}{m+1}, \frac{m}{m+1}\right)$ and $Q\left(\frac{3}{m+1}, \frac{3m}{m+1}\right)$

Hence equation of L_1 is

$$y - \frac{m}{m+1} = 2\left(x - \frac{1}{m+1}\right)$$

$\Rightarrow y - 2x - 1 = -\frac{3}{m+1}$ ... (1)

and that of L_2 is

$$y - \frac{3m}{m+1} = -3\left(x - \frac{3}{m+1}\right)$$

$\Rightarrow y + 3x - 3 = \frac{6}{m+1}$... (2)

From (1) and (2)

$$\frac{y-2x-1}{y+3x-3} = -\frac{1}{2}$$

$\Rightarrow x - 3y + 5 = 0$ which is a straight line.

Q.27. A straigh t lin e L with negative slope passes thr ough the point (8, 2) and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin. (2002 - 5 Marks)

Ans. 18

Sol. Let the equation of the line be (y – 2) = m (x – 8) where m < 0

$\Rightarrow \quad P \equiv \left(8 - \frac{2}{m}, 0\right)$ and $Q \equiv (0, 2 - 8m)$

Now, $OP + OQ = \left|8 - \frac{2}{m}\right| + |2 - 8m|$

$= 10 + \frac{2}{-m} + 8(-m) \geq 10 + 2\sqrt{\frac{2}{-m} \times 8(-m)} \geq 18$

Q.28. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through P (h, k) with the lines y = x and x + y = 2 is 4h^2. Find the locus of the point P. (2005 - 2 Marks)

Ans. y = 2x +1 or y = -2x +1

Sol. A line passing through P (h, k) and parallel to x-axis is

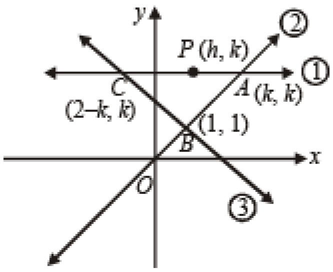
y = k. … (1)

The other two lines given are

y = x … (2)

and x + y = 2 … (3)

Let ABC be the Δ formed by the points of intersection of the lines (1), (2) and (3), as shown in the figure.



Then A (k, k), B (1, 1), C (2 – k, k)

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} k & k & 1 \\ 1 & 1 & 1 \\ 2-k & k & 1 \end{vmatrix} = 4h^2$

Operating $C_1 - C_2$ we get

$$\frac{1}{2}\begin{vmatrix} 0 & k & 1 \\ 0 & 1 & 1 \\ 2-2k & k & 1 \end{vmatrix} = 4h^2$$

$$\Rightarrow \frac{1}{2}|(2-2k)(k-1)| = 4h^2 \Rightarrow (k-1)^2 = 4h^2$$

$\Rightarrow k - 1 = 2h$ or $k - 1 = -2h$

$\Rightarrow k = 2h + 1$ or $k = -2h + 1$

$\therefore$ Locus of (h, k) is, $y = 2x + 1$ or $y = -2x + 1$.

Integar Type ques of Straight Lines and Pair of Straight Lines

Q.1. For a point P in the plane, let $d_1(P)$ a nd $d_2(P)$ be the distance of the point P from the lines x – y = 0 and x + y = 0 respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is (JEE Adv. 2014)

Ans. (6)

Sol. Let the point P be (x, y)

Then $d_1(P) = \left|\frac{x-y}{\sqrt{2}}\right|$ and $d_2(P) = \left|\frac{x+y}{\sqrt{2}}\right|$

For P lying in first quadrant x > 0, y > 0.

Also $2 \le d_1(P) + d_2(P) \le 4$

$$\Rightarrow 2 \le \left|\frac{x-y}{\sqrt{2}}\right| + \left|\frac{x+y}{\sqrt{2}}\right| \le 4$$

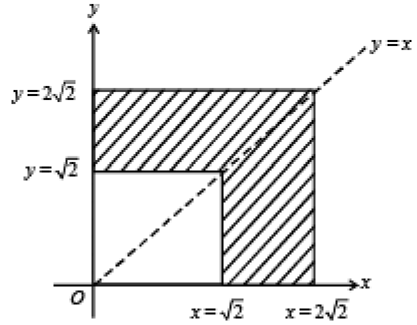
If x > y, then $2 \le \frac{x-y+x+y}{\sqrt{2}} \le 4$

or $\sqrt{2} \le x \le 2\sqrt{2}$

If x < y, then

$2 \le \frac{y-x+x+y}{\sqrt{2}} \le 4$ or $\sqrt{2} \le y \le 2\sqrt{2}$

The required region is the shaded region in the figure given below.



$\therefore$ Required area = $\left(2\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2 = 8 - 2 = 6$ sq units.